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Spectral dichotomy methods of a matrix pencil which calculate projectors on the subspace associated with the eigenvalues inside or outside of any circle, any ellipse or any parabola, were proposed. These methods are an extension of those proposed in [S. Traoré and M. Dosso, European Journal of Pure and Applied Mathematics, 15 (2), (2022) 681-725] and [S. Traoré, M. Dosso and L Samassi, International Journal of Numerical Methods and Applications, 22, (2022) 87-115] to matrix pencil.Before the presentation of our methods, a reminder of the spectral dichotomy methods of a matrix with respect to any circle, any ellipse or any parabola was made. Two numerical examples of matrix pencil show the good calculation of the projectors on the subspaces associated with the part of the plane concerned by the separation made by the figures.

Keywords and Phrases: spectral dichotomy method, spectral projector, eigen subspaces, eigenvalues.

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Spectral dichotomy methods of a matrix pencil which calculate projectors on the subspace associated with the eigenvalues inside or outside of any circle, any ellipse or any parabola, were proposed. These methods are an extension of those proposed in [S. Traoré and M. Dosso, European Journal of Pure and Applied Mathematics, 15 (2), (2022) 681-725] and [S. Traoré, M. Dosso and L Samassi, International Journal of Numerical Methods and Applications, 22, (2022) 87-115] to matrix pencil.Before the presentation of our methods, a reminder of the spectral dichotomy methods of a matrix with respect to any circle, any ellipse or any parabola was made. Two numerical examples of matrix pencil show the good calculation of the projectors on the subspaces associated with the part of the plane concerned by the separation made by the figures.

Keywords and Phrases : spectral dichotomy method, spectral projector, eigensubspaces, eigenvalues.

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I. INTRODUCTION

Spectral dichotomy methods were introduced by S. K. Godunov in [6] for the study of stability in the sense of Lyapunov. They consist in determining whether or not there are eigenvalues of a given matrix or a matrix pencil inside or outside a closed contour. If there are no eigenvalues in a neighborhood of the contour, these methods compute iteratevely the projector onto the invariant subspace associated to the eigenvalues of the matrix or the matrix pencil inside the contour. This computation is accompanied by the spectral norm $\|\mathbb{H}\|$ of a Hermitian positive definite matrix $\mathbb{H}$ called the dichotomy condition number. Initially developed on the imaginary axis by S. K. Godunov [6], these methods have been extended to the circle by Bulkgakov and Godunov in [2]. More efficient methods for computing the spectral projector and the dichotomy matrix in the circular case were proposed by Malyshev in [11, 12], Sadkane and Dosso in [4], Sadkane and Touhami in [17]. The methods of spectral dichotomy of a matrix or a matrix pencil with respect to the ellipse or parabola are respectively transformations of the methods with respect to the circle or imaginary axis. The methods with respect to the ellipse have been proposed by Godunov and Sadkane in [8, 9], Malyshev and sadkane in [14] and Sadkane and Touhami in [17] and those with respect to the parabola by Malyshev and sadkane in [14] and Sadkane and Touhami in [17]. Traoré et al. in [19, 20, 18] applied the dichotomy methods of a matrix with respect to the circle, the ellipse and the parabola all not centered at the origin.

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Let $zB - A$ be a regular matrix pencil with A and $B \in \mathbb{R}^{n\times n}(n > 1)$ with B non singular. The aim of this paper is to propose some methods of spectral dichotomy of the matrix pencil $zB - A$ with respect respectively to the circle $\mathcal{C}(\Omega, r)$ of center Ω with affix $\omega \neq 0$ and radius $r > 0$, the ellipse Ξ_{z_0} of equation

$$\left(\frac{x - x_0}{a}\right)^2 + \left(\frac{y - y_0}{b}\right)^2 = 1, \qquad a \geq b > 0 \tag{1.1}$$

and of center $z_0 = x_0 + iy_0$ different from 0 and the parabola of equation

$$2p(d - x) = (y - p\tilde{b})^2 \tag{1.2}$$

with $p > 0$.

Our work is organized as follows. In the section 2, we recall the spectral dichotomy methods of a matrix pencil with respect to the circle and the ellipse [16], all centered at the origin and with respect to the imaginary axis [16]. Algorithms developed in this section are useful for the sequel. In section 3, we propose new algorithms to compute the spectral projector and dichotomy matrix of a matrix pencil with respect to a circle, an ellipse or a parabola off centered at the origin. In this section, we use special matrix pencils to bring us back to the algorithms seen in the section 2. In section 4, we illustrate our results using graphs and tables to visualise and demonstrate the accuracy of our algorithms.

Throughout this work, we will use the following notation : the symbol $\|.\|$ denotes the 2-norm for vectors and matrices. The identity matrix (respectively the null matrix) of order k will be denoted by I_k (respectively 0_k) or just I (respectively 0) if the order is clear from the context.

II. SOME METHODS OF SPECTRAL DICHOTOMY OF A MATRIX PENCIL

Let $zB - A$ be a matrix pencil with $A,\ B \in \mathbb{R}^{n\times n}, (n > 1)$.

2.1 Method of spectral dichotomy of a matrix pencil with respect to a circle centered at the origin

Suppose that the matrix pencil $zB - A$ has no eigenvalue in a circle $\mathcal{C}(O, r)$, $(r > 0)$. Then

$$\mathbb{P} = \frac{1}{2\pi}\int_0^{2\pi}\left(B - e^{-i\theta}\frac{A}{r}\right)^{-1} B d\theta \tag{2.1}$$

is a spectral projector onto the subspace corresponding to the eigenvalues of $zB - A$ inside the circle $\mathcal{C}(O, r)$.

The numerical computation of $\mathbb{P}$ is accompanied by the dichotomy condition number given by the spectral norm of the Hermitian positive definite matrix

$$\mathbb{H} = \frac{1}{2\pi}\int_0^{2\pi}\left(B - e^{-i\theta}\frac{A}{r}\right)^{-*}\mathbb{H}^0\left(B - e^{-i\theta}\frac{A}{r}\right)^{-1} d\theta \tag{2.2}$$

where $\mathbb{H}^{(0)} = \mathbb{H}^{(0)^*} > 0$ is a given arbitrary matrix.

Consider the cyclic and finite linear system [8]

$$\begin{cases} BZ_0^{(2^j)} - \dfrac{A}{r} Z_1^{(2^j)} & = & I_n, \\ \\ BZ_k^{(2^j)} - \dfrac{A}{r} Z_{k+1}^{(2^j)} & = & 0, \qquad 1 \leq k \leq 2^j - 1. \end{cases} \tag{2.3}$$

From its solutions $Z_k^{(2^j)}$, the following theorem allows to obtain the values of $\mathbb{P}$ and $\mathbb{H}$ (see [8, 16]).

Theorem 2.1 *We have*

$$\mathbb{P} = \lim_{j \to +\infty} \mathbb{P}_j \qquad \textit{with} \quad \mathbb{P}_j = Z_{2j}^{(2^j)} B \equiv Z_0^{(2^j)} B \tag{2.4}$$

and

$$\mathbb{H} = \lim_{j \to +\infty} \mathbb{H}_j \qquad \textit{with} \quad \mathbb{H}_j = \sum_{k=1}^{2^j} (Z_k^{(2^j)})^* \mathbb{H}^0 (Z_k^{(2^j)}), \tag{2.5}$$

$$\mathbb{H}_j = V_j^* \mathbb{H}_{j-1} V_j + W_j^* \mathbb{H}_{j-1} W_j \tag{2.6}$$

where $V_j = BZ_{2^{j-1}}^{(2^j)} - \dfrac{A}{r} Z_1^{(2^j)}$ *and* $W_j = BZ_{2^j}^{(2^j)} - \dfrac{A}{r} Z_{2^{j-1}+1}^{(2^j)}$.

As for the computation of $Z_k^{(2^j)}$ and $\mathbb{H}_j$, they are explained in the following theorem (see [16])

Theorem 2.2 *Let*

$$Z_1^{(2)} = \Delta_0, \quad Z_2^{(2)} = \nabla_0$$

and for $j = 2, 3 \cdots$

$$Z_k^{(2^j)} = Z_k^{(2^{j-1})} \Delta_{j-1}, \quad k = 1, 2, \cdots 2^{j-1}.$$

$$Z_{k+2^{j-1}}^{(2^j)} = Z_{k+2^{j-1}}^{(2^{j-1})} \nabla_{j-1}, \quad k = 1, 2, \cdots 2^{j-1}.$$

and for $j = 2, 3 \cdots$

$$A_{j-1} + B_{j-1} = I_n \tag{2.7}$$

$$\Delta_{j-1} + \nabla_{j-1} = I_n \tag{2.8}$$

$$(2A_{j-1} - I_n)\Delta_{j-1} = A_{j-1} \tag{2.9}$$

with

$$A_{j-1} = -\frac{A}{r} Z_1^{(2^{j-1})}, \quad I_n = -Z_1^{(2^{j-1})}$$

and

$$H_j = \Delta_{j-1}^* H_{j-1} \Delta_{j-1} + (I_n - \Delta_{j-1})^* H_{j-1} (I_n - \Delta_{j-1}).$$

Following Algorithm 2.1 given in [17] summarizes the computations of the projector $\mathbb{P}$ and the dichotomy criterion $\mathbb{H}_j$.

Algorithm 2.1

Input : $A, B \in \mathbb{C}^{n\times n}$ *such the matrix pencil* $zB - A$ *is regular having no eigenvalues on the circle* $\mathcal{C}(O, r)$, $(r > 0)$.

$\mathbb{H}^{(0)} = \mathbb{H}^{(0)^*} > 0$ *used for scaling.*
Output : $\mathbb{P}$ *the projector onto the subspace of* $zB - A$ *associated with the eigenvalues inside the circle* $\mathcal{C}(O, r)$, $(r > 0)$.

$\mathbb{H}$ *the integral whose norm* $\|\mathbb{H}\|$ *indicates the numerical quality of the projector* $\mathbb{P}$.

1. *Initialisation and first iteration*

$$H_0 = H^0 \qquad \text{and} \quad \tilde{A} = \frac{A}{r}$$

- *Determine the solutions* X, Y *of equations*

$$X(B - \tilde{A}) = \tilde{A}, \quad Y(B - \tilde{A}) = I$$

- *Determine the solutions* Δ_0, ∇_0 *of equations*

$$(\tilde{A} + B)\Delta_0 = X, \quad (\tilde{A} + B)\nabla_0 = Y$$

- *compute* $H_1, Z_1^{(2)}, Z_2^{(2)}$*:*

$$H_1 = \Delta_0^* H_0 \Delta + \nabla_0^* H_0 \nabla_0$$

$$Z_1^2 = \Delta_0, \quad Z_2^{(2)} = \nabla_0$$

2. <u>*Iterations*</u> *for* $j = 2, 3 \cdots$

- *do*

$$A_{j-1} = -\tilde{A} Z_1^{2^{j-1}}$$

$$(2A_{j-1} - I_n)\Delta_{j-1} = A_{j-1}$$

- *Compute* $H_j, Z_1^{(2^j)}, Z_{2^j}^{(2^j)}$:

$$H_j = \Delta_{j-1}^* H_{j-1} \Delta_{j-1} + (I - \Delta_{j-1})^* H_{j-1}(I_n - \Delta_{j-1})$$

$$Z_1^{(2^j)} = Z_1^{(2^{j-1})}\Delta_{j-1}, Z_{2^j}^{(2^j)} = Z_{2^{j-1}}^{(2^{j-1})}(I_n - \Delta_{j-1})$$

end for

$$\mathbb{P}_0 = Z_{2^j}^{(2^j)} \qquad \text{and} \qquad \mathbb{H} = H_j$$

2.2 Method of spectral dichotomy of a matrix pencil with respect to an ellipse centered at the origin

Suppose that the matrix pencil $zB - A$ has no eigenvalues on the ellipse Ξ_0 of equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \qquad \text{with} \quad a \geq b > 0. \tag{2.10}$$

Consider the matrix pencil of the form $\mu\mathcal{B} - \mathcal{A}$ where

$$\mathcal{B} = \begin{pmatrix} \frac{a+b}{2}B & -\ A \\ 0 & \frac{a+b}{2}B \end{pmatrix} \qquad \text{and} \quad \mathcal{A} = \begin{pmatrix} -\frac{a-b}{2}B & 0 \\ A & -\frac{a-b}{2}B \end{pmatrix} \tag{2.11}$$

This following proposition shows the dichotomy parameters

$$\alpha = \sup_{z \in \Xi_{z_0}} \|(zI_n - A)^{-1}\| \qquad \text{and} \quad \widetilde{\alpha} = \sup_{|\widetilde{\lambda}|=1} \|(\widetilde{\lambda}\widetilde{\mathcal{B}} - \widetilde{\mathcal{A}})^{-1}\| \tag{2.12}$$

are equal.

Proposition 2.1 *Let α and $\widetilde{\alpha}$ defined in (3.5). then*

$$\alpha = \widetilde{\alpha}. \tag{2.13}$$

Moreover, if we denote by

$$\mathcal{P}_\infty = \begin{pmatrix} \widetilde{\mathcal{P}}_{11} & \widetilde{\mathcal{P}}_{12} \\ \widetilde{\mathcal{P}}_{21} & \widetilde{\mathcal{P}}_{22} \end{pmatrix}, \quad \text{with} \quad \widetilde{\mathcal{P}}_{ij} \in \mathbb{C}^{n\times n}, \quad i,j \in 1,2. \tag{2.14}$$

the projector onto the invariant subspace of the matrix pencil $\mu\mathcal{B} - \mathcal{A}$ associated with the eigenvalues inside $\mathcal{C}(O,r)$ and $\mathbb{P}_\infty \in \mathbb{C}^{n\times n}$ the projector onto the invariant subspace of the matrix pencil $zB - A$ associated with the eigenvalues inside Ξ_0.

The following proposition characterizes the relation between $\mathbb{P}_\infty$ and $\mathcal{P}_\infty$.

Proposition 2.2 *We have*

$$\mathbb{P}_\infty = \widetilde{\mathcal{P}}_{11} + \widetilde{\mathcal{P}}_{22}. \tag{2.15}$$

Following Algorithm 2.2 given in [17] summarizes the computations of the projector $\mathbb{P}_\infty$ and the dichotomy criterion $\mathbb{H}$.

Algorithm 2.2 .

- *Input variables: The matrices A and B, the parameters a, b such that the matrix pencil $zB - A$ has no eigenvalues on the ellipse Γ_0 of equation with $a \geq b > 0$.*
- *Output variables: $\mathbb{P}_\infty$ and $\mathbb{H}$*

  .

  $\mathbb{P}_\infty$ being the projector on the right invariant space of $zB - A$ corresponding to the eigenvalues outside the ellipse Γ_0 and $\mathbb{H}$ the dichotomy criterion.

1. *Set*

$$\mathcal{B} = \begin{pmatrix} \frac{a+b}{2}B & -A \\ 0 & \frac{a+b}{2}B \end{pmatrix} \quad \textit{and} \quad \mathcal{A} = \begin{pmatrix} -\frac{a-b}{2}B & 0 \\ A & -\frac{a-b}{2}B \end{pmatrix} \tag{2.16}$$

2. *Apply algorithm 2.1 to* $\lambda\mathcal{B} - \mathcal{A}$

3. *We obtain* $\mathbb{P}_\infty$ *and* $\mathbb{H}$.

2.3 Method of spectral dichotomy of a matrix pencil with respect to an imaginary axis

Suppose that the matrix pencil $zB - A$ has no eigenvalues on the imaginary axis. Consider the matrix pencil $\lambda\mathcal{B} - \mathcal{A}$ where $\mathcal{A} = A + B$ and $\mathcal{B} = A - B$. This following proposition shows the relation between the dichotomy parameters

$$\widetilde{\alpha} = \sup_{|\lambda|=1} \|\lambda\mathcal{B} - \mathcal{A})^{-1}\| \qquad \text{and} \quad \alpha = \sup_{\Re z=0} \|(zB - A)^{-1}\| \tag{2.17}$$

Proposition 2.3 *[17] Assume that* $\|A\| \leq 1$ *and let* α *and* $\widetilde{\alpha}$ *be the two parameters defined in (2.17)*

$$\frac{1}{2}\alpha \leq \widetilde{\alpha} \leq C(\alpha + 1) \tag{2.18}$$

with $C = \max(\|B\|, 1)$

Following Algorithm 2.3 given in [16, 17] summarizes the computations of the projector $\mathbb{P}_0$ and the dichotomy criterion $\mathbb{H}$.

Algorithm 2.3 *:*

- *Input variables :* A *and* B *such that the matrix pencil* $zB - A$ *has no eigenvalues on the imaginary axis.*
- *Output variables :* $\mathbb{P}_0$ *and* $\mathbb{H}$.
  $\mathbb{P}_0$ *being the projector on the invariant subspace of* $zB - A$ *corresponding to the eigenvalues of negative real parts and* $\mathbb{H}$ *the dichotomy criterion.*

1. *Determine the matrices* $\mathcal{A} = A + B$ *and* $\mathcal{B} = A - B$.

2. *Apply Algorithm 2.1 to* $\lambda\mathcal{B} - \mathcal{A}$.

3. $\mathbb{P}_0 = \mathcal{P}$.

III. SPECTRAL DICHOTOMY METHODS OF A MATRIX PENCIL WITH RESPECT TO AN OFF-CENTER CIRCLE, ELLIPSE OR PARABOLA

Let $zB - A$ be a matrix pencil with $A,\ B \in \mathbb{R}^{n\times n}(n > 1)$ such that B is not singular.

3.1 Spectral dichotomy method of a matrix pencil with respect to a circle not c entered at the origin

If the matrix pencil $zB - A$ has no eigenvalues on the circle $\mathcal{C}(\Omega, r); (r > 0)$, then the spectral projector on the invariant subspace corresponding to the eigenvalues on the disc $|z - b| < r$ is the matrix $\mathbb{P}_0$ defined by :

$$\begin{aligned}
\mathbb{P}_0 &= \frac{1}{2i\pi}\int_{\mathcal{C}(\Omega,r)} (zB - A)^{-1} B dz \\
&= \frac{1}{2i\pi}\int_{|z-b|=r} (zB - A)^{-1} B dz \\
&= \frac{1}{2i\pi}\int_0^{2\pi} \left((b + re^{i\theta})B - A\right)^{-1} Bire^{i\theta} d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left(B + \frac{bB - A}{r} e^{-i\theta}\right)^{-1} B d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left(B - \frac{\mathcal{A}}{r} e^{-i\theta}\right)^{-1} B d\theta
\end{aligned}$$

where $\mathcal{A} = A - bB$.

The computation of the spectral projector $\mathbb{P}$ is accompanied by that of a Hermitian matrix $\mathbb{H}$ defined by:

$$\mathbb{H} = H(r) = \frac{1}{2\pi}\int_0^{2\pi} \left(B - \frac{e^{-i\theta}\mathcal{A}}{r}\right)^{-*} H^{(0)} \left(B - \frac{e^{-i\theta}\mathcal{A}}{r}\right)^{-1} d\theta, \tag{3.1}$$

where $H^{(0)} = (H^{(0)})^* > 0$ is a positive definite Hermitian matrix used for setting purposes the scale.
We have the following algorithm that compute the projector $\mathbb{P}_0$ onto the invariant subspace of $zB - A$ corresponding to the eigenvalues inside the circle $\mathcal{C}(\Omega(b), r)$ and the Hermitian and definite positive matrix $\mathbb{H}$ associated.

Algorithm 3.1 .

- *Input variables : The matrices A and B, the parameters b and r such that the matrix pencil $zB - A$ has no eigenvalues on the circle $\mathcal{C}(\Omega, r)$.*
- *Output variables : $\mathbb{P}_0$ and $\mathbb{H}$.*
  $\mathbb{P}_0$ being the projector on the invariant space of $zB - A$ corresponding to the eigenvalues inside the circle $\mathcal{C}(\Omega(b), r)$ and $\mathbb{H}$ the matrix integral whose norm indicates the quality of the projector $\mathbb{P}_0$.

1. *Determine the matrix $\mathcal{A} = A - bB$.*

2. *Compute the projector $\mathbb{P}_0$ and the matrix $\mathbb{H}$ by applying Algorithm 2.1 to the matrix $\mathcal{A}$ and to the circle $\mathcal{C}(O, r)$ with center O and radius r.*

3.2 Spectral dichotomy method of a matrix pencil with respect to an ellipse not centered at the origin

Consider the ellipse

$$\Xi_{z_0} = \{z = x + iy \in \mathbb{C} \quad | \quad (x - x_0) + i(y - y_0) \in \Xi_0\} \tag{3.2}$$

of equation (1.1).

Suppose that the matrix pencil $zB - A$ has no eigenvalues on the ellipse Ξ_{z_0}. Consider the matrices of order $2n$ defined in the following way:

$$\widetilde{\mathcal{B}} = \begin{bmatrix} \frac{a+b}{2}B & -A_0 \\ 0 & \frac{a+b}{2}B \end{bmatrix} \quad \text{and} \quad \widetilde{\mathcal{A}} = \begin{bmatrix} -\frac{a-b}{2}B & 0 \\ A_0 & -\frac{a-b}{2}B \end{bmatrix} \tag{3.3}$$

where $A_0 = A - z_0 B$.

The eigenvalues $\widetilde{\lambda}$ of the matrix pencil $(\widetilde{\lambda}\widetilde{\mathcal{B}} - \widetilde{\mathcal{A}})$ and $\widetilde{z}$ of the matrix pencil $\widetilde{z}B - A_0$ are linked by the relation (3.4).

$$\widetilde{z} = \frac{(a+b)\widetilde{\lambda}^2 + (a-b)}{2\widetilde{\lambda}} \tag{3.4}$$

Consider the qualities of the spectral dichotomy with respect to the ellipse Ξ_{z_0} and the unit circle $\mathcal{C}(0,1)$ respectively defined by

$$\alpha = \sup_{z \in \Xi_{z_0}} \|(zB - A)^{-1} \quad \text{and} \quad \widetilde{\alpha} = \sup_{|\widetilde{\lambda}|=1} \|(\widetilde{\lambda}\widetilde{\mathcal{B}} - \widetilde{\mathcal{A}})^{-1}\| \tag{3.5}$$

Proposition 3.1 *Let α and $\widetilde{\alpha}$ be the quantities defined in (3.5). then*

$$\alpha = \widetilde{\alpha} \tag{3.6}$$

Proof

From (3.4), we have

$$\widetilde{z} = \frac{(a+b)\widetilde{\lambda}^2 + (a-b)}{2\widetilde{\lambda}}.$$

Which leads to

$$\frac{a+b}{2}B\widetilde{\lambda}^2 - A_0\widetilde{\lambda} + \frac{a-b}{2}B = \widetilde{\lambda}(\widetilde{z}B - A_0)$$

We have

$$(\mu^2\widetilde{\mathcal{B}} - \widetilde{\mathcal{A}})^{-1} = \begin{pmatrix} (\frac{a+b}{2}\mu^2 + \frac{a-b}{2})B & -\mu^2 A_0 \\ -A_0 & (\frac{a+b}{2}\mu^2 + \frac{a-b}{2})B \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \widetilde{z}\mu B & -\mu^2 A_0 \\ -A_0 & \widetilde{z}\mu B \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} I_n & 0 \\ 0 & \mu^{-1}I_n \end{pmatrix} \begin{pmatrix} \tilde{z}B & -A_0 \\ -A_0 & \tilde{z}B \end{pmatrix}^{-1} \begin{pmatrix} \mu^{-1}I_n & 0 \\ 0 & I_n \end{pmatrix}$$

$$= \begin{pmatrix} I_n & 0 \\ 0 & \mu^{-1}I_n \end{pmatrix} \times$$

$$\frac{1}{2} \begin{bmatrix} (\tilde{z}B - A_0)^{-1} + (\tilde{z}B + A_0)^{-1} & (\tilde{z}B - A_0)^{-1} - (\tilde{z}B + A_0)^{-1} \\ (\tilde{z}B - A_0)^{-1} - (\tilde{z}B + A_0)^{-1} & (\tilde{z}B - A_0)^{-1} + (\tilde{z}B + A_0)^{-1} \end{bmatrix}$$

$$\times \begin{pmatrix} \mu^{-1}I_n & 0 \\ 0 & I_n \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} I_n & 0 \\ 0 & \mu^{-1}I_n \end{pmatrix} \begin{pmatrix} \frac{I_n}{\sqrt{2}} & -\frac{I_n}{\sqrt{2}} \\ \frac{I_n}{\sqrt{2}} & \frac{I_n}{\sqrt{2}} \end{pmatrix}}_{K_1} \times \begin{bmatrix} (\tilde{z}B - A_0)^{-1} & 0 \\ 0 & (\tilde{z}B + A_0)^{-1} \end{bmatrix} \times$$

$$\underbrace{\begin{pmatrix} \frac{I_n}{\sqrt{2}} & \frac{I_n}{\sqrt{2}} \\ -\frac{I_n}{\sqrt{2}} & \frac{I_n}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \mu^{-1}I_n & 0 \\ 0 & I_n \end{pmatrix}}_{K_2}$$

We notice that the product $K_1K_2 = \mu^{-1}$. Therefore, with a double inequality, we deduce:

$$\sup_{z \in \Xi_0} \|(\tilde{z}B - A_0)^{-1}\| = \sup_{|\tilde{\lambda}|=1} \|(\tilde{\lambda}\widetilde{\mathcal{B}} - \widetilde{\mathcal{A}})^{-1}\|$$

We also know that $\tilde{z}B - A_0 = zB - A$, so we can conclude that

$$\sup_{z \in \Xi_{z_0}} \|(zB - A)^{-1}\| = \sup_{|\tilde{\lambda}|=1} \|(\tilde{\lambda}\widetilde{\mathcal{B}} - \widetilde{\mathcal{A}})^{-1}\|$$

whence the equality (3.6). $\square$

Moreover, if we denote by

$$\mathcal{P}_\infty = \begin{pmatrix} \widetilde{\mathcal{P}}_{11} & \widetilde{\mathcal{P}}_{12} \\ \widetilde{\mathcal{P}}_{21} & \widetilde{\mathcal{P}}_{22} \end{pmatrix} \qquad \text{with} \quad \mathcal{P}_{ij} \in \mathbb{C}^{n \times n}, i, j = 1, 2.$$

the projector associated with the eigenvalues of the matrix pencil $\lambda\mathcal{B} - \mathcal{A}$ outside the unit circle $\mathcal{C}(0, 1)$ and $\mathbb{P}_\infty$ the projector associated with the eigenvalues of the matrix pencil $zB - A$ inside Ξ_{z_0}.

The following proposition characterizes the link between $\mathbb{P}_\infty$ and $\mathcal{P}_\infty$.

Proposition 3.2 *we have*

$$\mathbb{P}_\infty = \widetilde{\mathcal{P}}_{11} + \widetilde{\mathcal{P}}_{22}$$

$$
\begin{bmatrix} \frac{a-b}{2}B & 0 \\ -A_0 & \frac{a-b}{2}B \end{bmatrix} \begin{bmatrix} \widetilde{X} & \widetilde{X}_1 \\ I_n & I_n \end{bmatrix}
$$

$$
+ \begin{bmatrix} \frac{a+b}{2}B & -A_0 \\ 0 & \frac{a+b}{2}B \end{bmatrix} \begin{bmatrix} \widetilde{X} & \widetilde{X}_1 \\ I_n & I_n \end{bmatrix} \begin{bmatrix} \widetilde{X}^2 & 0 \\ 0 & \widetilde{X}_1^2 \end{bmatrix}
$$

$$
= \begin{bmatrix} \dfrac{a-b}{2}\widetilde{X} & \dfrac{a-b}{2}\widetilde{X}_1 \\ -A_0\widetilde{X} + \dfrac{a-b}{2}I_n & -A_0\widetilde{X}_1 + \dfrac{a-b}{2}I_n \end{bmatrix} + \begin{bmatrix} (\dfrac{a+b}{2}\widetilde{X} - A_0)\widetilde{X}^2 & (\dfrac{a+b}{2}\widetilde{X}_1 - A_0)\widetilde{X}_1^2 \\ \dfrac{a+b}{2}\widetilde{X}^2 & \dfrac{a+b}{2}\widetilde{X}_1^2 \end{bmatrix}
$$

$$
= \begin{bmatrix} \dfrac{a-b}{2}\widetilde{X} + (\dfrac{a+b}{2}\widetilde{X} - A_0)\widetilde{X}^2 & \dfrac{a-b}{2}\widetilde{X}_1 + (\dfrac{a+b}{2}\widetilde{X}_1 - A_0)\widetilde{X}_1^2 \\ -A_0\widetilde{X} + \dfrac{a-b}{2}I_n + \dfrac{a+b}{2}\widetilde{X}^2 & -A_0\widetilde{X}_1 + \dfrac{a-b}{2}I_n + \dfrac{a+b}{2}\widetilde{X}_1^2 \end{bmatrix}
$$

$$
= \begin{bmatrix} (\dfrac{a+b}{2}\widetilde{X}^2 - A_0\widetilde{X} + \dfrac{a-b}{2})\widetilde{X} & (\dfrac{a+b}{2}\widetilde{X}_1^2 - A_0\widetilde{X}_1 + \dfrac{a-b}{2})\widetilde{X}_1 \\ \dfrac{a+b}{2}\widetilde{X}^2 - A_0\widetilde{X} + \dfrac{a-b}{2}I_n & \dfrac{a+b}{2}\widetilde{X}_1^2 - A_0\widetilde{X}_1 + \dfrac{a-b}{2}I_n \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
$$

Which leads to

$$
\begin{bmatrix} \frac{a+b}{2}I_n & -A_0 \\ 0 & \frac{a+b}{2}I_n \end{bmatrix}^{-1} \times \begin{bmatrix} \frac{a-b}{2}I_n & 0 \\ A_0 & \frac{a-b}{2}I_n \end{bmatrix} = \begin{bmatrix} \widetilde{X} & \widetilde{X}_1 \\ I_n & I_n \end{bmatrix} \times \begin{bmatrix} \widetilde{X}^2 & 0 \\ 0 & \widetilde{X}_1^2 \end{bmatrix} \times \begin{bmatrix} \widetilde{X} & \widetilde{X}_1 \\ I_n & I_n \end{bmatrix}^{-1}
$$

Let $\mathcal{P}_\infty$ be the projector on the unit circle associated with the eigenvalues of the matrix pencil $\lambda\widetilde{\mathcal{B}} - \widetilde{B}$ outside the circle.

$$
\mathcal{P}_\infty = \begin{pmatrix} \widetilde{X} & \widetilde{X}_1 \\ I_n & I_n \end{pmatrix} \begin{bmatrix} Q \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \widetilde{X} & \widetilde{X}_1 \\ I_n & I_n \end{pmatrix}^{-1}
$$

where k is the order of the matrix J_∞. Note that the matrix $X - X_1$ is nonsingular if and only if the matrix pencil $zI_n - A$ does not have $\pm\sqrt{a^2 - b^2}$ as own values. We can assume this without losing the generalities (the continuity arguments). Thereby

$$\mathcal{P}_\infty = \begin{pmatrix} \widetilde{X} & \widetilde{X}_1 \\ I_n & I_n \end{pmatrix} \begin{bmatrix} Q\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}Q^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} I_n & -\widetilde{X}_1 \\ -I_n & \widetilde{X} \end{pmatrix} \begin{pmatrix} \widetilde{X}-\widetilde{X}_1 & 0 \\ 0 & \widetilde{X}-\widetilde{X}_1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \begin{bmatrix} J_\infty & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_k & 0 & -\frac{a-b}{a+b}J_\infty^{-1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}^{-1}$$

$$\times \begin{pmatrix} \widetilde{X}-\widetilde{X}_1 & 0 \\ 0 & \widetilde{X}-\widetilde{X}_1 \end{pmatrix}$$

$$= \begin{bmatrix} Q\begin{pmatrix} J_\infty & 0 \\ 0 & 0 \end{pmatrix}Q^{-1} & Q\begin{pmatrix} -\frac{a-b}{a+b}I_k & 0 \\ 0 & 0 \end{pmatrix}Q^{-1} \\ Q\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}Q^{-1} & Q\begin{pmatrix} -\frac{a-b}{a+b}J_\infty^{-1} & 0 \\ 0 & 0 \end{pmatrix}Q^{-1} \end{bmatrix} \begin{pmatrix} \widetilde{X}-\widetilde{X}_1 & 0 \\ 0 & \widetilde{X}-\widetilde{X}_1 \end{pmatrix}^{-1}$$

$$\equiv \begin{pmatrix} \widetilde{\mathcal{P}}_{11} & \widetilde{\mathcal{P}}_{12} \\ \widetilde{\mathcal{P}}_{21} & \widetilde{\mathcal{P}}_{22} \end{pmatrix}.$$

with

$$\widetilde{\mathcal{P}}_{11} = Q\begin{bmatrix} J_\infty(J_\infty - \frac{a-b}{a+b}J_\infty^{-1})^{-1} & 0 \\ 0 & 0 \end{bmatrix}Q^{-1}$$

$$\widetilde{\mathcal{P}}_{12} = Q\begin{bmatrix} -\frac{a-b}{a+b}(J_\infty - \frac{a-b}{a+b}J_\infty^{-1})^{-1} & 0 \\ 0 & 0 \end{bmatrix}Q^{-1}$$

$$\widetilde{\mathcal{P}}_{21} = Q\begin{bmatrix} (J_\infty - \frac{a-b}{a+b}J_\infty^{-1})^{-1} & 0 \\ 0 & 0 \end{bmatrix}Q^{-1}$$

$$\widetilde{\mathcal{P}}_{22} = Q\begin{bmatrix} -\frac{a-b}{a+b}J_\infty^{-1}(J_\infty - \frac{a-b}{a+b}J_\infty^{-1})^{-1} & 0 \\ 0 & 0 \end{bmatrix}Q^{-1}$$

and we get $\widetilde{\mathcal{P}}_{11} + \widetilde{\mathcal{P}}_{22} = Q\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}Q^{-1} = \mathbb{P}_\infty$ $\square$

Let $\mathbb{H}$ the dichotomy criterion

$$\mathbb{H} = \frac{1}{4}\int_0^2 \left[(\frac{a+b}{2}e^{i\frac{\theta}{2}} + \frac{a-b}{2}e^{-i\frac{\theta}{2}})B - A_0\right]^{-*} \left[(\frac{a+b}{2}e^{i\frac{\theta}{2}} + \frac{a-b}{2}e^{-i\frac{\theta}{2}})B - A_0\right]^{-1} d\theta$$

Algorithm 3.2 .

- *Input variables: The matrices A and B, the parameters a, b and z_0 such that the matrix pencil $zB - A$ has no eigenvalues on the ellipse Ξ_0 of equation with $a \geq b > 0$.*

- *Output variables: $\mathbb{P}_\infty$ and $\mathbb{H}$.*
  $\mathbb{P}_\infty$ being the projector on the right invariant space of $zB - A$ corresponding to the eigenvalues outside the ellipse Ξ_{z_0} and $\mathbb{H}$ the criterion of dichotomy.

1. *Determine $A_0 = A - z_0 B$*
2. *Set*

$$\mathcal{B} = \begin{pmatrix} \frac{a+b}{2}B & -A_0 \\ 0 & \frac{a+b}{2}B \end{pmatrix} \quad \text{and} \quad \mathcal{A} = \begin{pmatrix} -\frac{a-b}{2}B & 0 \\ A_0 & \frac{a-b}{2}B \end{pmatrix} \tag{3.8}$$

2. *Apply Algorithm 2.2 to $\lambda\mathcal{B} - \mathcal{A}$*

3. *We obtain $\mathbb{P}_\infty$ et $\mathbb{H}$.*

3.3 Spectral dichotomy to a matrix pencil with respect to a parabola not centered at the origin

Let $zB - A$ be a regular matrix pencil of order n having no eigenvalues on the parabola Γ of equation

$$2p(d-x) = (y - p\tilde{b})^2 \tag{3.9}$$

with $p > 0$.

Let consider the following change of variable $\widetilde{x} = x + \frac{p}{2} - d$, we obtain

$$2p(\frac{p}{2} - \widetilde{x}) = (y - p\tilde{b})^2 \tag{3.10}$$

Consider the set
$$\widetilde{\Gamma}_d = \left\{ z = x + iy | \quad x + \left(\frac{p}{2} - d\right) + i(y - p\tilde{b}) \in \Gamma \right\}$$

described by the following equation (3.10).
We consider the following matrices of order $2n$

$$\widetilde{\mathcal{A}}_d = \begin{bmatrix} -\sqrt{\frac{p}{2}}B & A_{d\tilde{b}} \\ I_n & -\sqrt{\frac{p}{2}}I_n \end{bmatrix} \quad \text{with} \quad A_{db} = A + \left(\frac{p}{2} - d - ip\tilde{b}\right)B. \tag{3.11}$$

and

$$\mathcal{B} = \begin{bmatrix} B & 0 \\ 0 & I_n \end{bmatrix} \tag{3.12}$$

The respective eigenvalues $\widetilde{\lambda}_d$ and $z_{d\tilde{b}}$ of the matrices pencils $\lambda\mathcal{B} - \widetilde{\mathcal{A}}_d$ And $z_{d\tilde{b}}B - A_{d\tilde{b}}$ verifies the following relation

$$z_{d\tilde{b}} = \left(\sqrt{\frac{p}{2}} + \widetilde{\lambda}_d\right)^2.$$

$$\begin{aligned} z_{d\tilde{b}}B - A_{db} &= z_{d\tilde{b}}B - \left(A - \left(\frac{p}{2} - d - ip\tilde{b}\right) B\right) \\ &= z_{d\tilde{b}}B - A - \left(\frac{p}{2} - d - ip\tilde{b}\right) B \\ &= \left(z_{d\tilde{b}} - \left(\frac{p}{2} - d - ip\tilde{b}\right)\right) B - A \\ &= zB - A \end{aligned}$$

where z is the eigenvalue of the matrix pencil $zB - A$. Therefore,

$$z = \left(\sqrt{\frac{p}{2}} + \widetilde{\lambda}_d\right)^2 - \left(\frac{p}{2} - d - ip\tilde{b}\right) \tag{3.13}$$

We also assume that $\|A_{d\tilde{b}}\| = 1$. Otherwise (if $\|A_{d\tilde{b}}\| \neq 1$), we can take

$$A^1_{d\tilde{b}} = \frac{1}{\|A_{db}\|} A_{d\tilde{b}} \qquad \text{et} \qquad \tilde{p}^1_{db} = \frac{1}{\|A_{d\tilde{b}}\|}.$$

Let consider numerics parameters $\alpha_{\widetilde{\mathcal{A}}_d}$ et $\alpha_{A_{d\tilde{b}}}$ defined by

$$\alpha_{\widetilde{\mathcal{A}}_d} = \sup_{\Re(\widetilde{\lambda}_d)=0} \|(\widetilde{\lambda}_d \mathcal{B} - \widetilde{\mathcal{A}}_d)^{-1}\| \;\; \text{and} \;\; \alpha_{A_{d\tilde{b}}} = \sup_{z \in \widetilde{\Gamma}_d} \|(zB - A)^{-1}\| \tag{3.14}$$

The following proposition gives a relation between the parameters $\alpha_{\widetilde{\mathcal{A}}_d}$ et $\alpha_{A_{d\tilde{b}}}$.

Proposition 3.3 *Assume that* $\max(\|A\|, \|B\|) \leq 1$ *and let* $\alpha_{\widetilde{\mathcal{A}}_d}$ *and* $\alpha_{A_{d\tilde{b}}}$ *be the two parameters defined in* (3.18). *Suppose that*

$$\|A_{d\tilde{b}}\| = 1 \qquad \textit{and} \qquad \left|\frac{p}{2} - d - ip\tilde{b}\right| < \frac{1}{\alpha_{A_{d\tilde{b}}}} \tag{3.15}$$

then

$$\alpha_{A_{d\tilde{b}}} \leq \alpha_{\widetilde{\mathcal{A}}_d} \leq C\left(\alpha_{A_{d\tilde{b}}} + \sqrt{\alpha_{A_{d\tilde{b}}}}\left(\sqrt{1 + \alpha_{A_{d\tilde{b}}}} + 1\right)\right). \tag{3.16}$$

with $C = \max(\|B^{-1}\|, 1)$.

Proof

Let the matrix

$$\left(\widetilde{\lambda}_d \mathcal{B} - \widetilde{\mathcal{A}}_d\right) = \begin{bmatrix} (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})B & -A_{d\tilde{b}} \\ -I_n & (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})I_n \end{bmatrix}$$

We have

$$\begin{bmatrix} (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})I_n & A_{d\tilde{b}} \\ I_n & (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})B \end{bmatrix} \times \begin{bmatrix} (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})B & -A_{d\tilde{b}} \\ -I_n & (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})I_n \end{bmatrix} =$$

$$\begin{bmatrix} (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})^2 B - A_{d\tilde{b}} & 0 \\ 0 & (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})^2 B - A_{d\tilde{b}} \end{bmatrix}$$

thus

$$(\widetilde{\lambda}_d\mathcal{B} - \widetilde{\mathcal{A}}_d)^{-1} = \begin{bmatrix} (\widetilde{\lambda}_d+\sqrt{\frac{p}{2}})^2B - A_{db} & 0 \\ 0 & (\widetilde{\lambda}_d+\sqrt{\frac{p}{2}})^2B - A_{d\tilde{b}} \end{bmatrix}^{-1} \times \begin{bmatrix} (\widetilde{\lambda}_d+\sqrt{\frac{p}{2}})B & A_{d\tilde{b}} \\ I_n & (\widetilde{\lambda}_d+\sqrt{\frac{p}{2}})I_n \end{bmatrix}$$

$$= \begin{bmatrix} (\widetilde{\lambda}_d+\sqrt{\frac{p}{2}})^2B - (A+(\frac{p}{2}-d-ip\tilde{b})B) & 0 \\ 0 & (\widetilde{\lambda}_d+\sqrt{\frac{p}{2}})^2B - (A+(\frac{p}{2}-d-ip\tilde{b})B) \end{bmatrix}^{-1} \times \begin{bmatrix} (\widetilde{\lambda}_d+\sqrt{\frac{p}{2}})B & (A+(\frac{p}{2}-d-ip\tilde{b})B) \\ I_n & (\widetilde{\lambda}_d+\sqrt{\frac{p}{2}})I_n \end{bmatrix}$$

$$= \begin{bmatrix} ((\widetilde{\lambda}_d+\sqrt{\frac{p}{2}})^2 - \frac{p}{2} + d - ip\tilde{b})B - A & 0 \\ 0 & ((\widetilde{\lambda}_d+\sqrt{\frac{p}{2}})^2 - \frac{p}{2} + d - ip\tilde{b})B - A \end{bmatrix}^{-1} \times \begin{bmatrix} (\widetilde{\lambda}_d+\sqrt{\frac{p}{2}})I_n & A+(\frac{p}{2}-d-ip\tilde{b})B) \\ I_n & (\widetilde{\lambda}_d+\sqrt{\frac{p}{2}})I_n \end{bmatrix}$$

$$= \begin{bmatrix} (zB-A)^{-1} & 0 \\ 0 & (zB-A)^{-1} \end{bmatrix} \times \begin{bmatrix} \sqrt{z+\frac{p}{2}-d-ip\tilde{b}}I_n & A+(\frac{p}{2}-d-ip\tilde{b})B \\ I_n & \sqrt{z+\frac{p}{2}-d-ip\tilde{b}}I_n \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{z+\frac{p}{2}-d-ip\tilde{b}}(zB-A)^{-1} & (zB-A)^{-1}(A+(\frac{p}{2}-d-ip\tilde{b})B) \\ (zB-A)^{-1} & \sqrt{z+\frac{p}{2}-d-ipb}(zB-A)^{-1} \end{bmatrix}$$

Knowing that the norm of $(\widetilde{\lambda}_d I_{2n} - \widetilde{\mathcal{A}}_d)^{-1}$ is greater than or equal to the norm of each of its block components taken individually, we can deduce that

$$\alpha_{\widetilde{\mathcal{A}}_d} = \sup_{\Re(\widetilde{\lambda}_d)=0} \|(\widetilde{\lambda}_d\mathcal{B} - \widetilde{\mathcal{A}}_d)^{-1}\| \geq \sup_{z\in\widetilde{\Gamma}_d} \|(zB-A)^{-1}\| = \alpha_{A_{d\tilde{b}}}$$

and also

$$\left\|(\widetilde{\lambda}_d\mathcal{B}-\widetilde{\mathcal{A}}_d)^{-1}\right\| \leq \left\|\begin{pmatrix} \sqrt{z+\frac{p}{2}-d-ip\tilde{b}}I_n & A+(\frac{p}{2}-d-ip\tilde{b})B \\ I_n & \sqrt{z+\frac{p}{2}-d-ip\tilde{b}}I_n \end{pmatrix}\right\| \|(zB-A)^{-1}\|$$

$$\leq \left\|\begin{pmatrix} \|\sqrt{z+\frac{p}{2}-d-ip\tilde{b}}I_n\| & \|A+(\frac{p}{2}-d-ip\tilde{b})B\| \\ \|I_n\| & \|\sqrt{z+\frac{p}{2}-d-ip\tilde{b}}I_n\| \end{pmatrix}\right\| \|(zB-A)^{-1}\|$$

$$\leq \left\|\begin{pmatrix} \sqrt{|z|}+\sqrt{|\frac{p}{2}-d-ip\tilde{b}|} & 1 \\ 1 & \sqrt{|z|}+\sqrt{|\frac{p}{2}-d-ip\tilde{b}|} \end{pmatrix}\right\| \|(zB-A)^{-1}\|$$

$$\leq \left(\sqrt{|z|}+\sqrt{|\frac{p}{2}-d-ip\tilde{b}|}+1\right) \|(zB-A)^{-1}\|$$

- if $|z| \leq \dfrac{\alpha_{A_{d\tilde{b}}}+1}{\alpha_{A_{d\tilde{b}}}}$ then

$$\begin{aligned}\|(\lambda\mathcal{B}-\widetilde{\mathcal{A}}_d)^{-1}\| &\leq \alpha_{A_{d\tilde{b}}}(\sqrt{|z|}+\sqrt{|\frac{p}{2}-d-ip\tilde{b}|}+1)\\ &\leq \alpha_{A_{d\tilde{b}}}\sqrt{\frac{\alpha_{A_{d\tilde{b}}}+1}{\alpha_{A_{d\tilde{b}}}}+\frac{1}{\sqrt{\alpha_{d\tilde{b}}}}+1\Big)}\\ &\leq \alpha_{A_{d\tilde{b}}}+\sqrt{\alpha_{d\tilde{b}}}(\sqrt{\alpha_{d\tilde{b}}+1}+1)\end{aligned}$$

- Assume that $|z| > \dfrac{\alpha_{A_{d\tilde{b}}}+1}{\alpha_{A_{d\tilde{b}}}}$, we get

$$\begin{aligned}(zB-A)(zB-A)^{-1} &= I_n\\ (zB)(zB-A)^{-1}-A(zB-A)^{-1} &= I_n\\ (zB)(zB-A)^{-1} &= I_n+A(zB-A)^{-1}I_n\\ (zB-A)^{-1} &= (zB)^{-1}(I_n+A(zB-A)^{-1})\end{aligned}$$

As a result

$$\left\|(\widetilde{\lambda}_d\mathcal{B}-\widetilde{\mathcal{A}}_d)^{-1}\right\| \leq \left\|(zB)^{-1}(I_n+A(zB-A)^{-1})\right\| \times \left\|1+\sqrt{|z|}+\sqrt{\left|\frac{p}{2}-d-ip\tilde{b}\right|}\right\|$$

$$\leq \left\|B^{-1}\right\|\left(\left\|A(zB-A)^{-1}\right\|+1\right)\frac{1+\sqrt{|z|}+\sqrt{\left|\frac{p}{2}-d-ip\tilde{b}\right|}}{|z|}$$

$$\leq C\left(\|A\|\left\|(zB-A)^{-1}\right\|+1\right)\frac{1+\sqrt{|z|}+\sqrt{\left|\frac{p}{2}-d-ip\tilde{b}\right|}}{|z|}$$

$$\leq C\left(\alpha_{A_{d\tilde{b}}}+1\right)\left(\frac{1}{\sqrt{|z|}}+\frac{1}{|z|}+\frac{\sqrt{|\frac{p}{2}-d-ip\tilde{b}|}}{|z|}\right)$$

$$\leq C\left(\alpha_{A_{d\tilde{b}}}+1\right)\quad\frac{\sqrt{\alpha_{A_{d\tilde{b}}}}}{\sqrt{\alpha_{A_{d\tilde{b}}}+1}}+\frac{\alpha_{A_{d\tilde{b}}}}{\alpha_{A_{d\tilde{b}}}+1}+\frac{\sqrt{\alpha_{A_{d\tilde{b}}}}}{\alpha_{A_{d\tilde{b}}+1}}\Bigg)$$

$$\leq C\left(\sqrt{\alpha_{d\tilde{b}}+1}\sqrt{\alpha_{d\tilde{b}}}+\alpha_{d\tilde{b}}+\sqrt{\alpha_{d\tilde{b}}}\right)$$

$$\leq C\left(\alpha_{A_{d\tilde{b}}}+\sqrt{\alpha_{A_{d\tilde{b}}}}\left(\sqrt{\alpha_{A_{d\tilde{b}}+1}}+1\right)\right)$$

Finally

$$\alpha_{A_{d\tilde{b}}} \leq \alpha_{\mathcal{A}_{d\tilde{b}}} \leq C\left(\alpha_{A_{d\tilde{b}}}+\sqrt{\alpha_{A_{d\tilde{b}}}}\left(\sqrt{\alpha_{A_{d\tilde{b}}+1}}+1\right)\right).$$

Moreover, if we denote by

$$\widetilde{\mathcal{P}}_{+}=\begin{pmatrix}\widetilde{\mathcal{P}}_{11}^{(d)} & \widetilde{\mathcal{P}}_{12}^{(d)}\\ \widetilde{\mathcal{P}}_{21}^{(d)} & \widetilde{\mathcal{P}}_{22}^{(d)}\end{pmatrix}\qquad \text{avec}\qquad \widetilde{\mathcal{P}}_{ij}^{(d)}\in\mathbb{C}^{n\times n},\qquad i,j=1,2. \tag{3.17}$$

the projector onto the subspace of $\widetilde{\mathcal{A}}_d$ associated with the eigenvalues in the left complex half-plane and $\widetilde{\mathbb{P}}_\infty\in\mathbb{C}^{n\times n}$ the projector onto the suspace of $zB-A_{d\tilde{b}}$ associated with the eigenvalues inside the parabola $\widetilde{\Gamma_d}$, then the following proposition characterizes the link between $\widetilde{\mathbb{P}}_\infty$ and $\widetilde{\mathcal{P}}_+$

Proposition 3.4 *We have*

$$\widetilde{\mathbb{P}}_\infty=2\widetilde{\mathcal{P}}_{11}^{(d)}=2\widetilde{\mathcal{P}}_{22}^{(d)}=4\widetilde{\mathcal{P}}_{12}^{(d)}\widetilde{\mathcal{P}}_{21}^{(d)} \tag{3.18}$$

Moreover

$$\widetilde{\mathbb{P}}_\infty A=4(\widetilde{\mathcal{P}}_{12}^{(d)})^2-(p-2d-2ipb)B\widetilde{\mathcal{P}}_{11}^{(d)} \tag{3.19}$$

Proof

Let $\widetilde{X_d}$ be a solution of the matrix equation

$$\left(\tilde{X}_d + \sqrt{\frac{p}{2}} I_n\right)^2 = A_{d\tilde{b}}. \tag{3.20}$$

Let $\tilde{X}_1$ be another solution of the same matrix equation (3.20). We have

$$\begin{aligned}
&\left(\tilde{X}_d + \sqrt{\frac{p}{2}} I_n\right)^2 = \left(\tilde{X}_1 + \sqrt{\frac{p}{2}} I_n\right)^2 \\
&\Leftrightarrow \left(\tilde{X}_d + \sqrt{\frac{p}{2}} I_n\right) = \left(\tilde{X}_1 + \sqrt{\frac{p}{2}} I_n\right) \qquad \text{where} \qquad \left(\tilde{X}_d + \sqrt{\frac{p}{2}} I_n\right) = -\left(\tilde{X}_1 + \sqrt{\frac{p}{2}} I_n\right) \\
&\Leftrightarrow \tilde{X}_d = \tilde{X}_1 \qquad \text{where} \quad \tilde{X}_1 = -\tilde{X}_d - 2\sqrt{\frac{p}{2}} I_n
\end{aligned}$$

Finally the other solution $\tilde{X}_1$ is written in the form $\tilde{X}_1 = -\tilde{X} - 2\sqrt{\frac{p}{2}} I_n$. We get [19]

$$\tilde{\mathcal{A}}_d = \begin{bmatrix} \tilde{X}_d + \sqrt{\frac{p}{2}} I_n & -\tilde{X}_d - \sqrt{\frac{p}{2}} I_n \\ I_n & I_n \end{bmatrix} \times \begin{bmatrix} \tilde{X}_d & 0 \\ 0 & -\tilde{X}_d - 2\sqrt{\frac{p}{2}} I_n \end{bmatrix} \times \begin{bmatrix} \frac{1}{2}(\tilde{X}_d + \sqrt{\frac{p}{2}} I_n)^{-1} & \frac{1}{2} I_n \\ -\frac{1}{2}(\tilde{X}_d + \sqrt{\frac{p}{2}} I_n)^{-1} & \frac{1}{2} I_n \end{bmatrix}$$

Soit $\tilde{X}_d = Q_{d\tilde{b}} \begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} Q_{db}^{-1}$ the canonical jordan form of the matrix $\tilde{X}_d$ with M_+ and M_- the Jordan blocks associated respectively with the eigenvalues of $\tilde{X}_d$ located in the right half-plane and the left half-plane.

By replacing $\tilde{X}_d$ by its decomposition in the matrix $\tilde{\mathcal{A}}_d$, we get

$$\widetilde{\mathcal{A}}_d = \widetilde{\mathcal{Q}}_{d\tilde{b}} \mathcal{M} (\widetilde{\mathcal{Q}}_{d\tilde{b}})^{-1}$$

with

$$\widetilde{\mathcal{Q}_{d\tilde{b}}} = \begin{bmatrix} Q_{d\tilde{b}} & 0 \\ 0 & Q_{d\tilde{b}} \end{bmatrix} \begin{bmatrix} \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix} + \sqrt{\frac{p}{2}} I_n & -\begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} - \sqrt{\frac{p}{2}} I_n \\ I_n & I_n \end{bmatrix}$$

and

$$\mathcal{M} = \begin{bmatrix} \begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} - 2\sqrt{\frac{p}{2}} I_n \end{bmatrix}$$

We can therefore compute the associated projector

$$
\begin{aligned}
\widetilde{\mathcal{P}}_{+} &= (\widetilde{\mathcal{Q}}_{d\tilde{b}}) \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} (\widetilde{\mathcal{Q}}_{d\tilde{b}})^{-1} \\
&= \begin{bmatrix} Q_{d\tilde{b}} & 0 \\ 0 & Q_{d\tilde{b}} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} + \sqrt{\frac{p}{2}} I_n & -\begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} - \sqrt{\frac{p}{2}} I_n \\ I_n & I_n \end{bmatrix} \times \begin{bmatrix} \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \\
&\times \begin{bmatrix} \frac{1}{2} \left(\begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} + \sqrt{\frac{p}{2}} I_n \right)^{-1} & \frac{1}{2} I_n \\ -\frac{1}{2} \left(\begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} - \sqrt{\frac{p}{2}} I_n \right)^{-1} & \frac{1}{2} I_n \end{bmatrix} \begin{bmatrix} Q_{d\tilde{b}}^{-1} & 0 \\ 0 & Q_{d\tilde{b}}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} Q_{d\tilde{b}} & 0 \\ 0 & Q_{d\tilde{b}} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} M_+ + \sqrt{\frac{p}{2}} I_k & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \\
&\times \begin{bmatrix} \frac{1}{2} \begin{bmatrix} (M_+ + \sqrt{\frac{p}{2}} I_k)^{-1} & 0 \\ 0 & (M_- + \sqrt{\frac{p}{2}} I_{n-k})^{-1} \end{bmatrix} & \frac{1}{2} \begin{bmatrix} I_k & 0 \\ 0 & I_{n-k} \end{bmatrix} \\ -\frac{1}{2} \begin{bmatrix} (M_+ + \sqrt{\frac{p}{2}} I_k)^{-1} & 0 \\ 0 & (M_- + \sqrt{\frac{p}{2}} I_{n-k})^{-1} \end{bmatrix} & \frac{1}{2} \begin{bmatrix} I_k & 0 \\ 0 & I_{n-k} \end{bmatrix} \end{bmatrix} \begin{bmatrix} Q_{d\tilde{b}}^{-1} & 0 \\ 0 & Q_{d\tilde{b}}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} Q_{d\tilde{b}} & 0 \\ 0 & Q_{d\tilde{b}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} & \frac{1}{2} \begin{bmatrix} M_+ + \sqrt{\frac{p}{2}} I_k & 0 \\ 0 & 0 \end{bmatrix} \\ \frac{1}{2} \begin{bmatrix} (M_+ + \sqrt{\frac{p}{2}} I_k)^{-1} & 0 \\ 0 & 0 \end{bmatrix} & \frac{1}{2} \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} Q_{d\tilde{b}}^{-1} & 0 \\ 0 & Q_{d\tilde{b}}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} Q_{d\tilde{b}} \begin{bmatrix} \frac{1}{2} I_k & 0 \\ 0 & 0 \end{bmatrix} Q_{d\tilde{b}}^{-1} & Q_{db} \begin{bmatrix} \frac{1}{2} (M_+ + \sqrt{\frac{p}{2}} I_k) & 0 \\ 0 & 0 \end{bmatrix} Q_{d\tilde{b}}^{-1} \\ Q_{d\tilde{b}} \begin{bmatrix} \frac{1}{2} (M_+ + \sqrt{\frac{p}{2}} I_k)^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q_{d\tilde{b}}^{-1} & Q_{d\tilde{b}} \begin{bmatrix} \frac{1}{2} I_k & 0 \\ 0 & 0 \end{bmatrix} Q_{d\tilde{b}}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} \widetilde{\mathcal{P}}_{11}^{(d)} & \widetilde{\mathcal{P}}_{12}^{(d)} \\ \widetilde{\mathcal{P}}_{21}^{(d)} & \widetilde{\mathcal{P}}_{22}^{(d)} \end{bmatrix}
\end{aligned}
$$

It follows that

$$\widetilde{\mathcal{P}}_{11}^{(d)} = Q_d \begin{bmatrix} \frac{1}{2}I_k & 0 \\ 0 & 0 \end{bmatrix} Q_{d\tilde{b}}^{-1} = \frac{1}{2}\widetilde{\mathbb{P}}_\infty$$

$$\widetilde{\mathcal{P}}_{12}^{(d)} = \frac{1}{2}Q_{d\tilde{b}} \begin{bmatrix} M_+ + \sqrt{\frac{p}{2}}I_k & 0 \\ 0 & 0 \end{bmatrix} Q_{d\tilde{b}}^{-1}$$

$$\widetilde{\mathcal{P}}_{21}^{(d)} = \frac{1}{2}Q_{db} \begin{bmatrix} (M_+ + \sqrt{\frac{p}{2}}I_k)^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q_{d\tilde{b}}^{-1}$$

$$\widetilde{\mathcal{P}}_{22}^{(d)} = Q_{d\tilde{b}} \begin{bmatrix} \frac{1}{2}I_k & 0 \\ 0 & 0 \end{bmatrix} Q_{d\tilde{b}}^{-1} = \frac{1}{2}\widetilde{\mathbb{P}}_\infty$$

with $\widetilde{X_d} = Q_{d\tilde{b}} \begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} Q_{d\tilde{b}}^{-1}$ we have

$$\begin{aligned} A &= A_{d\tilde{b}} - \left(\frac{p}{2} - d - ipb\right) B \\ &= Q_{d\tilde{b}} \begin{bmatrix} \left(M_+ + \sqrt{\frac{p}{2}}I_k\right)^2 - \left(\frac{p}{2} - d - ip\tilde{b}\right) B_1 & -\left(\frac{p}{2} - d - ip\tilde{b}\right) B_2 \\ -\left(\frac{p}{2} - d - ip\tilde{b}\right) B_3 & \left(M_- + \sqrt{\frac{p}{2}}I_{n-k}\right)^2 - \left(\frac{p}{2} - d - ip\tilde{b}\right) B_4 \end{bmatrix} Qj_{d\tilde{b}}^{-1} \end{aligned}$$

where $B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$ and

$$\begin{aligned} \widetilde{\mathbb{P}}_\infty A &= Q_{d\tilde{b}} \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \left(M_+ + \sqrt{\frac{p}{2}}I_k\right)^2 - \left(\frac{p}{2} - d - ip\tilde{b}\right) B_1 & -\left(\frac{p}{2} - d - ip\tilde{b}\right) B_2 \\ -\left(\frac{p}{2} - d - ip\tilde{b}\right) B_3 & \left(M_- + \sqrt{\frac{p}{2}}I_{n-k}\right)^2 - \left(\frac{p}{2} - d - ip\tilde{b}\right) B_4 \end{bmatrix} Q_{d\tilde{b}}^{-1} \\ &= Q_{db} \begin{bmatrix} \left(M_+ + \sqrt{\frac{p}{2}}I_k\right)^2 - (\frac{p}{2} - d - ip\tilde{b})B_1 & 0 \\ 0 & 0 \end{bmatrix} Q_{d\tilde{b}}^{-1} \\ &= 4(\widetilde{\mathcal{P}}_{12}^{(d)})^2 - (p - 2d - 2ip\tilde{b})B\widetilde{\mathcal{P}}_{11}^{(d\tilde{b})} \end{aligned}$$

so (3.18) et (3.19). □

Remark 3.1 *We note that:*

- *If the parameter $d = \frac{p}{2}$, hence the equalities (3.19) are reduced to $\widetilde{\mathbb{P}}_\infty A = 4(\widetilde{\mathcal{P}}_{12}^{(d)})^2 + 2ipbB\widetilde{\mathcal{P}}_{11}^{(d)}$.*
- *If the parameter $b = 0$, hence the equalities(3.19) are reduced to $\widetilde{\mathbb{P}}_\infty A = 4(\widetilde{\mathcal{P}}_{12}^{(d)})^2 - (p - 2d)B\widetilde{\mathcal{P}}_{11}^{(d)}$.*
- *If the parameters $b = 0, d = \frac{p}{2}$, hence the equalities (3.19) are reduced to $\widetilde{\mathbb{P}}_\infty A = 4(\widetilde{\mathcal{P}}_{12}^{(d)})^2$.*

Algorithm 3.3 (DichoP)

- *Input variables : A and B such that the matrix pencil $zB-A$ has no eigenvalues on the parabola with equation* $2p\left(\frac{p}{2}-\tilde{x}\right)^2=(y-pb)^2$ *with* $p>0$

- *Output variables:* $\mathbb{P}$ *and* $\mathbb{H}$

  $\mathbb{P}$ *being the projector on the right subspace of* $zB-A$ *associated with the eigenvalues outside the parabola and the matrix* $\mathbb{H}$ *whose norm gives the dichotomy criterion.*

1. - *If* $d=\frac{p}{2}$ *and* $b=0$, *set*

$$\mathcal{A}=\begin{pmatrix}-\sqrt{\frac{p}{2}}B & A\\ I_n & -\sqrt{\frac{p}{2}}I_n\end{pmatrix}\qquad \text{and}\quad \mathcal{B}=\begin{pmatrix}B & 0\\ 0 & I_n\end{pmatrix}$$

   - *elseif* $d=\frac{p}{2}$ *and* $b\neq 0$, *set*

$$\mathcal{A}=\begin{pmatrix}-\sqrt{\frac{p}{2}}B & A-ipbB\\ I_n & -\sqrt{\frac{p}{2}}I_n\end{pmatrix}\qquad \text{and}\quad \mathcal{B}=\begin{pmatrix}B & 0\\ 0 & I_n\end{pmatrix}$$

   - *elseif* $d\neq\frac{p}{2}$ *and* $b=0$, *set*

$$\mathcal{A}=\begin{pmatrix}-\sqrt{\frac{p}{2}}B & A+(\frac{p}{2}-d)B\\ I_n & -\sqrt{\frac{p}{2}}I_n\end{pmatrix}\qquad \text{and}\quad \mathcal{B}=\begin{pmatrix}B & 0\\ 0 & I_n\end{pmatrix}$$

   - *else* $d\neq\frac{p}{2}$ *and* $b\neq 0$, *set*

$$\mathcal{A}=\begin{pmatrix}-\sqrt{\frac{p}{2}}B & A+(\frac{p}{2}-d-ip\tilde{b})B\\ I_n & -\sqrt{\frac{p}{2}}I_n\end{pmatrix}\qquad \text{and}\quad \mathcal{B}=\begin{pmatrix}B & 0\\ 0 & I_n\end{pmatrix}$$

2. *Using Algorithm 2.3 to* $\lambda\mathcal{B}-\mathcal{A}$, *compute the Projector* $\mathcal{P}$ *onto the right eigenspace of* $\mathcal{A}$ *associated with the eigenvalues on the right half-plane of the complex plane and the matrix* $\mathbb{H}_{d\tilde{b}}$ *;*

3. *If* $\|\mathbb{H}\|$ *is not large, determine the projectors* $\mathbb{P}$ *by*

$$\mathbb{P}=2\widetilde{\mathcal{P}}_1$$

IV. NUMERICAL TESTS

Example 4.1 *We consider the matrix pencil $zB-A$ of order 20 defined by [18]*

$$A=\begin{cases} a_{2i,2i}=-\dfrac{1}{3} & if \quad 1\le i\le 10\\ a_{2i-1,2i+1}=1 & if \quad 1\le i\le 9\\ a_{2i+1,2i-1}=\dfrac{1}{3} & if \quad 1\le i\le 9\\ a_{2i,2i+1}=3 & if \quad 1\le i\le 9,\\ a_{i,j}=0 & otherwise\end{cases}\quad \text{and}\quad B=\begin{cases} b_{2i-1,2i-1}=-\dfrac{1}{4} & if \quad 1\le i\le 10\\ b_{2i,2i+2}=-1 & if \quad 1\le i\le 9\\ b_{2i+2,2i}=5 & if \quad 1\le i\le 9\\ b_{2i,2i-1}=2 & if \quad 1\le i\le 9,\\ b_{i,j}=0 & otherwise\end{cases}\tag{4.1}$$

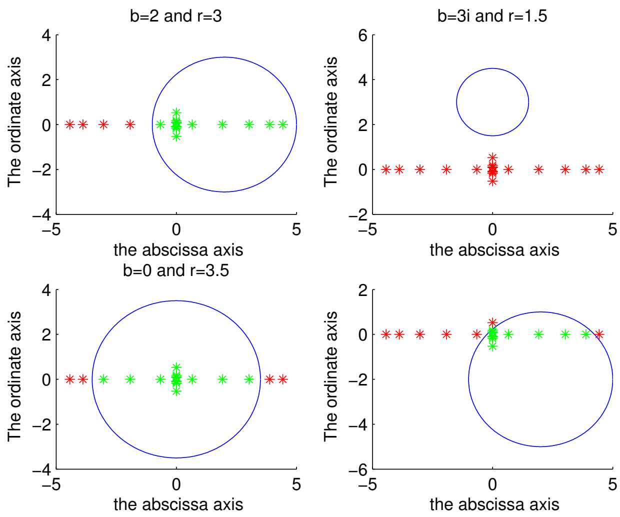

Figure 1: Partition of the spectrum of the matrix pencil $zB - A$ by the off-center circle with center with affix $z0$.

Table 1: Traces, norms and quality of the spectral projector $\mathbb{P}_0$ by applying Algorithm 3.1 for four values of the couple (b, r)

(b, r)	$tr(\mathbb{P}_0)$	$\|\|\mathbb{P}_0\|\|$	$\|\|\mathbb{P}_0^2 - \mathbb{P}_0\|\|$	$\|\|\mathbb{P}_0 W - W\mathbb{P}_0\|\|$	$\|\|\mathbb{H}_0\|\|$
$(2, 3)$	16	$1.9428\ 10^{04}$	$4.4207\ 10^{-10}$	$4.9178\ 10^{-7}$	$6.5654\ 10^{10}$
$(3i, 1.5)$	0	0	0	0	$4.2800\ 10^{08}$
$(0, 3.5)$	16	$1.9089\ 10^{4}$	$1.5983\ 10^{-11}$	$4.6416\ 10^{-6}$	$6.6210\ 10^{10}$
$(2-2i, 3)$	13	$2.1814\ 10^{4}$	$5.3396\ 10^{-9}$	$3.3987\ 10^{-7}$	$8.8018\ 10^{10}$

Table 2: Traces, norms and quality of the spectral projector $\mathbb{P}_\infty$ by applying Algorithm 3.2 for values $a, b, z0$

.

$(a, b, z0)$	$tr(\mathbb{P}_\infty)$	$\|\|\mathbb{P}_\infty\|\|$	$\|\|\mathbb{P}_\infty^2 - \mathbb{P}_\infty\|\|$	$\|\|\mathbb{P}_\infty W - W\mathbb{P}_\infty\|\|$	$\|\|\mathbb{H}_\infty\|\|$
$(6, \sqrt(3), 1)$	20	1	$7.7410\ 10^{-14}$	$2.8899\ 10^{-12}$	$1.3900\ 10^{8}$
$(2, \sqrt(2), i)$	11	$2.3358\ 10^{4}$	$1.5964\ 10^{-7}$	$1.0407\ 10^{-6}$	$5.0865\ 10^{10}$
$(4, 1, 1-i)$	5	$1.6850\ 10^{4}$	$5.4046\ 10^{-7}$	$2.8924\ 10^{-6}$	$3.9620\ 10^{11}$
$(3, 1.2, 2+2i)$	0	$3.0725\ 10^{-8}$	$3.0725\ 10^{-8}$	$1.4260\ 10^{-7}$	$1.9655\ 10^{9}$

Table 3: Traces, norms and quality of the spectral projector $\mathbb{P}$ by applying Algorithm 3.3 for four values p, b and d.

(p, b, d)	$tr(\mathbb{P})$	$\|\|\mathbb{P}\|\|$	$\|\|\mathbb{P}^2 - \mathbb{P}\|\|$	$\|\|\mathbb{P}W - W\mathbb{P}\|\|$	$\|\|\mathbb{H}\|\|$
$(2, 0, 10)$	20	1	$4.8955\ 10^{-13}$	$1.0086\ 10^{-12}$	$4.2198\ 10^{6}$
$(1.5, 3, 1)$	16	$2.7889\ 10^{-8}$	$2.7889\ 10^{-8}$	$2.0312\ 10^{-8}$	$1.7304\ 10^{10}$
$(1, 4, 1)$	0	$3.5172\ 10^{-9}$	$3.5172\ 10^{-9}$	$2.6957\ 10^{-8}$	$3.5087\ 10^{8}$
$(3, 2, 3)$	3	$1.8947\ 10^{4}$	$1.4058\ 10^{-7}$	$1.3711\ 10^{-7}$	$9.0937\ 10^{11}$

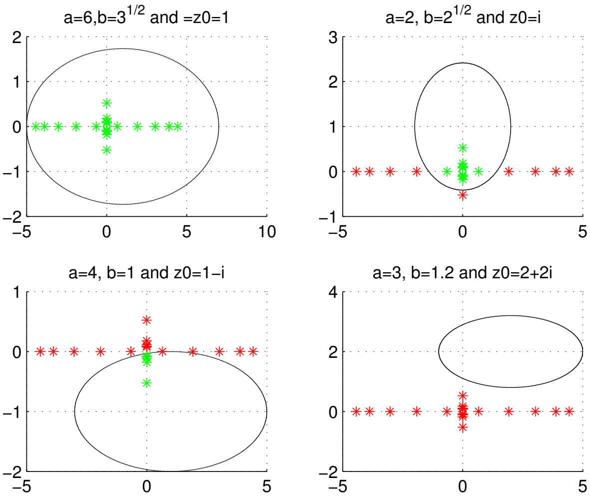

Figure 2: Partition of the spectrum of the matrix pencil $zB - A$ by the off-center ellipse with center with affix $z0$ and parameters a and b.

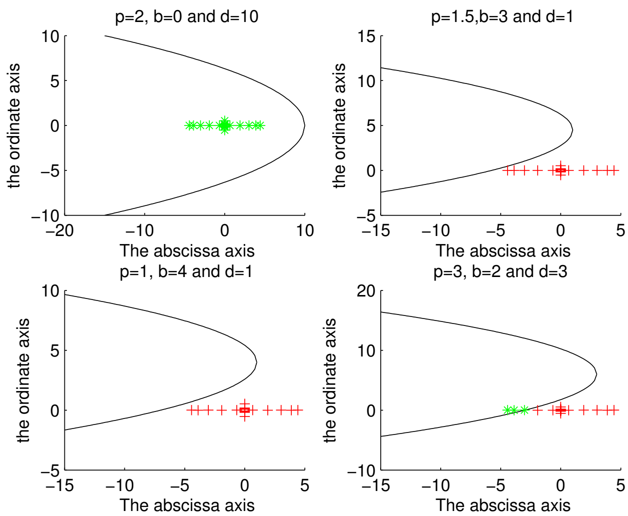

Figure 3: Partition of the spectrum of the matrix pencil $zB - A$ by the parabola of equation $2p(d-x) = y^2$.

The graphs above effectively show a dichotomy of the spectrum of the matrix pencil by the various curves which are the circle, the ellipse and the parabola all not centered at the origin. To corroborate our algorithms, we give in tables, the values of the norms of the projector $\mathbb{P}$ *and of the norm of the dichotomy matrix* $\mathbb{H}$.

We note that the values taken by the norms of the projector $\mathbb{P}$ *and of its dichotomy criterion* $\mathbb{H}$ *are high . This is due to the fact that the matrices* A *and* B *are badly conditioned (*$\kappa(A) = 2.836310^3$ *et* $\kappa(B) = 3.253410^4$ *where* $\kappa(A) = \|A\|.\|A^-1\|$*).*

Example 4.2 *Consider the matrices*

$$A = \begin{bmatrix} 0.49 & 0.89 & 0.33 & 0.32 & 1.09 & -0.01 \\ 1.03 & -1.15 & -0.75 & 0.31 & 1.11 & 1.53 \\ 0.73 & -1.07 & 1.37 & -0.86 & -0.86 & -0.77 \\ -0.30 & -0.81 & -1.71 & -0.03 & 0.08 & 0.37 \\ 0.29 & -2.94 & -0.10 & -0.16 & -1.21 & -0.23 \\ -0.79 & 1.44 & -0.24 & 0.63 & -1.11 & 1.12 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 1.5 & 0 & 0 & 1 & 0 \\ 0 & 4 & 1 & 1 & 0 & 1 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.2}$$

with matrix B a no singular matrix.

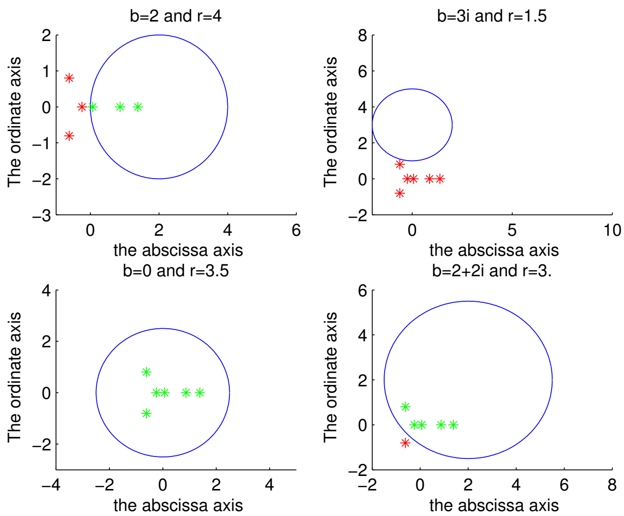

Figure 4: Partition of the spectrum of the matrix pencil$zB - A$ by the off-center circle with center with affix $z0$.

Table 4: Traces, norms and quality of the spectral projector $\mathbb{P}_0$ by applying Algorithm 3.1 for four values of the couple (b, r)

(b, r)	$tr(\mathbb{P}_0)$	$\|\mathbb{P}_0\|$	$\|\mathbb{P}_0^2 - \mathbb{P}_0\|$	$\|\mathbb{P}_0 W - W\mathbb{P}_0\|$	$\|\mathbb{H}_0\|$
$(2, 4)$	6	1	$3.1317\ 10^{-16}$	$4.4730\ 10^{-16}$	1.3256
$(3i, 1.5)$	0	0	0	0	0.4696
$(0, 3.5)$	6	1	$3.4282\ 10^{-16}$	$7.9911\ 10^{-16}$	1.2442
$(2 + 2i, 3)$	4	1.8397	$5.8237\ 10^{-16}$	$4.3141\ 10^{-15}$	68.5791

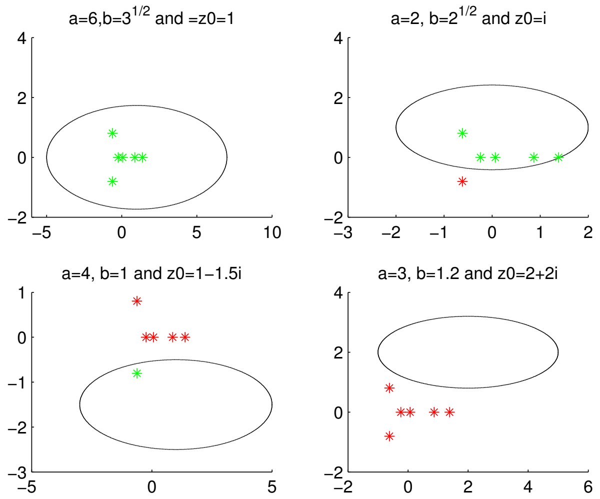

Figure 5: Partition of the spectrum of the matrix pencil$zB - A$ by the off-center ellipse with center with affix $z0$ and parameters a and b.

Table 5: Traces, norms and quality of the spectral projector $\mathbb{P}_\infty$by applying Algorithm 3.2 for values $a, b, z0$.

$(a, b, z0)$	$tr(\mathbb{P}_\infty)$	$\|\mathbb{P}_\infty\|$	$\|\mathbb{P}_\infty^2 - \mathbb{P}_\infty\|$	$\|\mathbb{P}_\infty W - W\mathbb{P}_\infty\|$	$\|\mathbb{H}_\infty\|$
$(6, \sqrt{(3)}, 1)$	6	1	$6.6926\ 10^{-16}$	$8.9554\ 10^{-16}$	0.1147
$(2, \sqrt{(2)}, i)$	5	1.7637	$8.2272\ 10^{-16}$	$8.3327\ 10^{-15}$	17.3309
$(4, 1, 1 - 1.5i)$	1	1.7637	$1.1051\ 10^{-15}$	$8.9629e - 15\ 10^{-15}$	0.3607
$(3, 1.2, 2 + 2i)$	0	$1.8846\ 10^{-15}$	$1.8846\ 10^{-15}$	$3.9832\ 10^{-15}$	0.3056

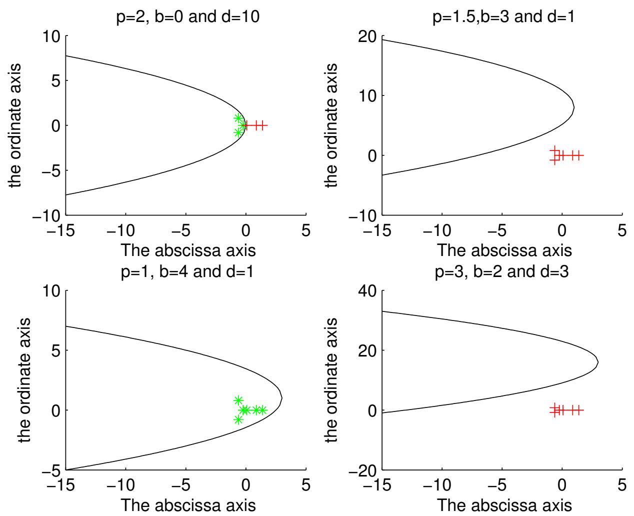

Figure 6: Partition of the spectrum of the matrix pencil$zB - A$ by the parabola of equation $2p(d-x) = y^2$.

Table 6: Traces, norms and quality of the spectral projector $\mathbb{P}$ by applying Algorithm 3.3 for four values p, b and d.

(p,b,d)	$tr(\mathbb{P})$	$\|\mathbb{P}\|$	$\|\mathbb{P}^2-\mathbb{P}\|$	$\|\mathbb{P}W-W\mathbb{P}\|$	$\|\mathbb{H}\|$
$(2,0,10)$	6	1	$4.5305e-16\ 10^{-16}$	$3.9732\ 10^{-16}$	1.4152
$(1.5,3,1)$	0	$5.0943\ 10^{-15}$	$5.0943\ 10^{-15}$	$4.8198\ 10^{-15}$	1.3944
$(1,4,1)$	0	0	$2.6692\ 10^{-15}$	$6.6636\ 10^{-15}$	1.2688
$(3,2,3)$	0	$5.5735\ 10^{-15}$	$5.5735\ 10^{-15}$	$8.3372\ 10^{-15}$	8.7003

The graphs above effectively show a dichotomy of the spectrum of the matrix pencil by the various curves which are the circle, the ellipse and the parabola all not centered at the origin. To corroborate our algorithms, we give in tables, the values of the norms of the projector $\mathbb{P}$ and of the norm of the dichotomy matrix $\mathbb{H}$.

We can notice that the values of the norm of $\mathbb{H}$ are small. This proves the very good quality of the dichotomy carried out

V. CONCLUSION

After a few reminders on methods of spectral dichotomy of a pencil matrix with respect to a circle, an ellipse or a parabola all not centered at the origin and/or not symmetrical with respect to the abscissa axis, we presented, by making changes of variables, methods of spectral dichotomy of a pencil matrix with respect to an off-center circle, an off-center ellipse or any parabola. In the spectral dichotomy method of a pencil matrix $zB - A$ with respect to

- a circle $\mathcal{C}(\Omega, r)$, $(r > 0)$ not originally centered, by considering the change of variable $\mathcal{A} = A - bB$, we obtain projector $\mathbb{P}_0$ on the invariant space of the pencil matrix corresponding to the eigenvalues inside the circle and a Matrix $\mathbb{H}$ whose the norm indicates the quality of the projector, by applying the spectral dichotomy method of the pencil matrix $zB - \mathcal{A}$ with respect to the circle $\mathcal{C}(O, r)$ centered at the origin.
- an ellipse Ξ_{z_0} centered at a point $z_0 \neq 0$, by considering the change of variable $A_0 = A - z_0B$, we obtain the projector $\mathbb{P}_\infty$ on the right invariant space of $zB - A$ corresponding to the eigenvalues outside the ellipse Ξ_{z_0} and a Matrix $\mathbb{H}$ whose the norm indicates the quality of the projector, by applying the spectral dichotomy method of a pencil matrix $zB - A_0$ with respect to the ellipse Ξ_0 centered at the origin.
- any parabola Γ of equation $2p(d-x) = (y - p\tilde{b})^2$ with $p > 0$, by considering the change of variable $A_{db} = A + \left(\frac{p}{2} - d - ip\tilde{b}\right)B$, we obtain $\mathbb{P}$ on the right subspace of $zB - A$ associated with the eigenvalues outside the parabola and a Matrix $\mathbb{H}$ whose the norm indicates the quality of the projector, by applying the spectral dichotomy method of a pencil matrix $zB - A_{db}$ with respect to the parabola $2p\left(\frac{p}{2} - x\right) = y^2$ studied by Malyshev and Sadkane.

Numerical tests on two examples of matrices pencils show that the presented methods compute well the projectors if there are no eigenvalues on or in a close neighborhood of the figures.

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